first in a positive direction, and after training show a little increase of positive asymmetry, with the exception of the strength-weight index, in which the skewness decreases to nearly perfect symmetry.

The relation between capacity for modification and the initial position in the scale can be determined only after calculation of coefficients of correlation, and for this purpose correlation tables are now being constructed.

The relation between the amount of modification and the length of time of training has been studied in only one series of measurements, that of the strength of legs. The measurements were plotted for every second month, that is, October, December, February, April and June. The magnitude of the mean was found to increase during each succeeding period, rapidly at first and then more and more slowly. The increase amounted during the first period to 20 kilos, during the second to 8.8 kilos, third to 6.4 kilos, and fourth to 0.37 kilos.

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*SOME FUNDAMENTAL DISCOVERIES IN MATHEMATICS.**

THE oldest extensive work on mathematics which has been deciphered was written by an Egyptian named Ahmes between 1700 and 2000 B.C. It bears the following title: 'Direction for obtaining a knowledge of all dark things * * * all secrets which are contained in the things,' and claims to be modeled after writings which were then old. The first part is devoted to a table in which every fraction whose numerator is 2 and whose denominator is any odd number from 5 to 99 is resolved into the sum of fractions with unity as a common numerator. The following are examples:

$$\tfrac{2}{5}=\tfrac{1}{3}+\tfrac{1}{15},\ \tfrac{2}{7}=\tfrac{1}{4}+\tfrac{1}{28},\ \tfrac{2}{19}=\tfrac{1}{12}+\tfrac{1}{76}+\tfrac{1}{114}.$$

As this table is constructed according to no general rule, it is probable that it is a collection of results obtained by mathematicians during a long period of years. In fact some of these numbers are found in a mathematical papyrus which is many hundred years older than the work of Ahmes. This table, therefore, furnishes one of the many evidences of the fact that the early development of mathematics is largely based upon experiments. Comprehensive rules and theorems are a much later product.

From a modern point of view it might be said that the theory of arithmetical progression marked the highest point reached by Ahmes in arithmetic. He solves linear algebraic equations involving one unknown and considers the area of a circle equivalent to a square whose side is eight ninths of the diameter. This is equivalent to calling $\pi = 3.1605$, which is a much closer approximation than many later nations employed.* To find the area of an isosceles triangle he multiplied the base by half of one of the equal sides instead of by half the altitude. This inaccuracy seems to be due to the fact that the Egyptians did not know how to extract the square root of a number, and hence they could not find the exact area of such a triangle from its sides.

While the work of Ahmes is of the greatest interest to the mathematical historian, yet it contains few facts of sufficient generality or beauty to be classed among the fundamental discoveries in mathematics. It emphasizes rules rather than thought. In fact, it is practically confined to problems and answers, with the verifications of

* Read before the Science Association of Stanford University, November 5, 1902.

* Cf. I. Kings, ch. 7, v. 23 and II. Chronicles, ch. 4, v. 2.

some of the answers. It appears that the Egyptian did not make any additions to the work of Ahmes for more than a thousand years.

About the seventh century B.C. the Greeks showed such a deep interest in learning that they began to go to foreign countries (especially Egypt) in quest of knowledge. They soon excelled their teachers, and inaugurated a golden period of mathematical progress which has had no equal until recent times. Hence we naturally look to the Greeks for fundamental discoveries whose beauty and generality have engaged the admiration and interest of all who became acquainted with them.

One of the earliest of these is the proof that there are lines which do not have a common measure. Pythagoras observed that it is impossible to divide the side of a square into such a number of equal parts that one of these parts is contained an integral number of times in the diagonal. This fact made a very deep impression on the Greek mind. It is one of those great truths which can never be fully established by experiment, and yet does not rest on postulates or axioms which appear somewhat arbitrary. It thus stands in sharp contrast with the older discovery that the sum of the angles of a plane triangle is equal to two right angles, and deserves to be placed in a higher category of mathematical truths.

A scholium of Euclid's 'Elements,' which is supposed to be due to Proclus, bears evidence of the high regard which the Greeks had for the discovery of the incommensurable or the irrational. It reads as follows: "It is said that the man who first made the theory of the irrational public died in a shipwreck because the unspeakable and invisible should always be kept secret, and that he who by chance first touched and uncovered this symbol of life was removed to the origin of things where the eternal waves wash around him. Such is the reverence in which these men held the theory of the irrational quantities."

Aristotle frequently speaks of the fact that the diagonal of a square whose side is unity is irrational, and in one instance states that otherwise an even number would be equal to an odd number. The meaning of this is made clear by Euclid's proof of the fact that the $\sqrt{2}$, which is the value of the given diagonal, is irrational. Euclid says, in substance, if we assume that $\sqrt{2} = m/n$ a rational number, it follows that $2n^2 = m^2$. The fraction m/n may be supposed to be reduced to its lowest terms, and hence at least one of the two numbers m, n must be odd. This, however, makes the equation $2n^2 = m^2$ impossible, since the square of an even number is always divisible by 4 and the square of an odd number is odd. By dividing both members of the equation $2n^2 = m^2$ by 2 an odd number would be equal to an even number, as Aristotle says. It appears very probable that this elegant proof is due to the Pythagoreans, possibly to Pythagoras himself.

Another fundamental discovery of the Greeks is the use of infinite convergent series. Aristotle observed that the sum of an infinite number of small things may be finite and Archimedes frequently finds the sum of an infinite series in the solution of a problem. For instance, in finding the area of a portion of the parabola he observes that it is equal to the area of a given triangle multiplied by the infinite geometric series $1 + \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \cdots$ and he proves that the sum of this series cannot be greater or less than 4/3. His proof is practically the same as that found in our elementary algebras.

In finding the sum of such infinite series Archimedes answered in a very explicit and definite manner some of the difficult questions raised by Zeno two hundred years

earlier. For instance, Zeno argued that it was impossible to go from one place to another, because one would have to go one half the distance before arriving, but before going half the distance one would have to go one half of this half and so on to infinity. He also argued that Achilles could not overtake a tortoise which moved at one tenth his rate because by the time Achilles reached the place where the tortoise had been when he started the tortoise would have moved some distance ahead, and by the time Achilles reached this spot the tortoise would again have moved some distance ahead, and so on to infinity. These difficulties were completely solved by the Greek mathematicians, and further serious arguments along this line seemed to be based upon ignorance or perversity.

The Greeks were the greatest mathematicians of antiquity and Archimedes was the greatest mathematician among the Greeks. It is, therefore, of especial interest to learn what Archimedes himself regarded as his highest achievements. These consist of several important theorems in regard to the sphere, viz., that the volume of a sphere is two thirds of the volume of the circumscribed cylinder, and the area of the sphere is two thirds of the area of this cylinder. The beauty of these theorems impressed Archimedes so forcibly that he requested that a sphere inscribed in a cylinder should be marked on his tombstone. It is well known that Cicero discovered the grave of Archimedes by means of this inscription.

With the two theorems just mentioned Archimedes classed his closely related theorem, that the area of a zone with one base is equal to that of a circle whose radius is the distance from the base of the zone to its pole. These theorems may have appealed to Archimedes more forcibly on account of the fact that the Pythagoreans used to say that the sphere was the most beautiful of the solids and the circle the most beautiful of the plane figures. There are, however, few theorems in elementary mathematics which establish such unexpected and important facts.

The Greeks studied mathematics for its own sake. They cared little about the practical applications of their results. The following story about Euclid is characteristic: "A youth who had begun to read geometry with Euclid when he had learned the first proposition inquired, 'What do I get by learning those things?' So Euclid called his slave and said, 'Give him three oboli, since he must gain out of what he learns.'" The maxim of the Pythagoreans, 'A figure and step forwards, not a figure to gain three oboli,' is evidence of the same spirit.

In their disinterested search for truth they incidentally made more progress in practical results than was made by other nations who had these results directly in view. The fact that their extensive developments of the conic sections had to wait nearly two thousand years until they found application in the astronomical theories of Kepler, Newton and others is frequently cited as evidence of the importance of developing knowledge for its own sake.

Notwithstanding the remarkable achievements of the ancient Greeks we have to look to a less noted people for one of the most fundamental discoveries of elementary arithmetic, viz., the use of the zero. If we think how cumbersome arithmetic operations become when no use can be made of the zero, it may appear to us marvelous that Europe should have learned the use of this number symbol less than a thousand years ago.

At the last international congress of mathematicians the leading mathematical

historian, Moritz Cantor, expressed the opinion that the use of zero was probably due to the Babylonians 1700 B.C. However, it has not been definitely established that zero was in use any earlier than 400 A.D. About this time it was used in India, and several centuries later the Arabs began to employ it. Through the Arabs its use became known to Europeans during the twelfth century. It was not generally adopted in Europe until several centuries later, notwithstanding its great advantages. For a considerable time there were two parties among the European educators—one party, known as the algorists, favored the adoption of the Hindu system of notation (falsely called Arabic) with its position values, while the other, known as the abacists, favored the Roman notation without zero or position value.

The general adoption of the Hindu system was greatly facilitated by the facts that it was explained in most of the calendars for more than a century beginning with 1300, and that the medieval universities frequently offered courses devoted to the use of this notation. With the opening of the medieval universities we approach some of the fundamental discoveries in more modern mathematics. As we considered these on a similar occasion,* we shall merely add a few thoughts on the concept of dimensions which are due to Plücker.

The idea of more than three dimensions can be partially explained in a very simple manner. If the total number of points on a straight line is denoted by ∞ (the symbol for infinity), it is clear that there are ∞^2 points in a plane, since through each point of the given line we may draw a line at right angles to this line. Each of these ∞^2 points of the plane may be taken as the center of an infinite number of circles, and all the circles which have one point as center are distinct from those which have any other point as center. Hence there are ∞^3 circles in a plane, while there are only ∞^2 points in it.

* SCIENCE, Vol. XL. (1900), p. 528.

We arrive at the same result by observing that an infinite number of lines may be drawn through each point of a plane and that each of these lines is tangent to an infinite number of circles going through this point. Hence ∞^2 circles pass through each point of a plane and lie entirely in the plane. As the number of points on a circle is infinite, the number of circles is obtained by multiplying the number of points by ∞. Hence we say that the plane is two-dimensional when the point is considered as the element, but it is three-dimensional if the circle is considered as element. If the ellipse were taken as element it could be readily shown that the plane would be five-dimensional.

Similarly space is three-dimensional if the point is taken as element but it is four-dimensional if the sphere is taken as element. Since there are ∞^6 pairs of points in space and ∞^2 pairs of points on a line there are ∞^4 lines in space, that is there is a 1, 1 correspondence between the lines and spheres of space. This is frequently expressed by saying there are just as many spheres in our space as there are lines, while the number of each of these is infinitely larger than the number of points. From this standpoint there is no limit to the number of dimensions of ordinary space.

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SCIENTIFIC BOOKS.

The Yucceæ. By WILLIAM TRELEASE. From the Thirteenth Annual Report of the Missouri Botanical Garden. Issued July 30, 1902. St. Louis, Mo. Published by the Board of Trustees. 1902. 8vo. Pp. 107.

The Spanish bayonets are shrubby or tree-like plants, principally of the genus *Yucca*, and represented in gardens by short-stemmed